Solution for HW8.

Ex16.8: 14) Let $\rho = \sqrt{x^2+y^2+z^2}$.

Then $\frac{\partial}{\partial x}\left(\frac{x}{\rho}\right) = \frac{1}{\rho} + x\left(-\frac{1}{2}\right)\rho^{-3}(2x) = \frac{1}{\rho} - \frac{x^2}{\rho^3}$

Similarly, $\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$, $\frac{\partial}{\partial z}\left(\frac{z}{\rho}\right) = \frac{1}{\rho} - \frac{z^2}{\rho^3}$.

$\Rightarrow \nabla\cdot F = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{2}{\rho}$

$\Rightarrow \text{Flux} = \iiint_D \frac{2}{\rho}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \left(\frac{2}{\rho}\right)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\,d\phi\,d\theta = 2\pi\int_0^{\pi} 3\sin\phi\,d\phi = 12\pi$ //

16) $\frac{\partial}{\partial x}(\ln(x^2+y^2)) = \frac{2x}{x^2+y^2}$, $\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}$

$= -\frac{2z}{x^2+y^2}$, $\frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2}$.

$\Rightarrow \nabla\cdot F = \frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2}$.

$\Rightarrow \text{Flux} = \iiint_D \left(\frac{2x-2z}{x^2+y^2} + \sqrt{x^2+y^2}\right) dz\,dy\,dx$

$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2}\left(\frac{2r\cos\theta - 2z}{r^2} + r\right) r\,dz\,dr\,d\theta$

$= 2\pi\int_1^{\sqrt{2}}\int_{-1}^{2}\left(\frac{-2z}{r} + r^2\right) dz\,dr$

$= 2\pi\int_1^{\sqrt{2}}\left(-\frac{3}{r} + 3r^2\right) dr$

$= 2\pi\left(-3\ln\sqrt{2} + 2\sqrt{2} - 1\right)$ //

20) By symmetry, we only consider the ith-component of both sides.

a) let $F_1 = (P_1, Q_1, R_1)$, $F_2 = (P_2, Q_2, R_2)$.

$$F_1 \times F_2 = \begin{vmatrix} i & j & k \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix}$$

$$= (Q_1R_2 - Q_2R_1,\ P_2R_1 - P_1R_2,\ P_1Q_2 - P_2Q_1)$$

$$\nabla \times (F_1 \times F_2) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q_1R_2 - Q_2R_1 & P_2R_1 - P_1R_2 & P_1Q_2 - P_2Q_1 \end{vmatrix}$$

$\therefore$ i-th component of $\nabla \times (F_1 \times F_2)$

$$= \frac{\partial P_1}{\partial y} Q_2 + P_1 \frac{\partial Q_2}{\partial y} - \frac{\partial P_2}{\partial y} Q_1 - P_2 \frac{\partial Q_1}{\partial y} - \frac{\partial P_2}{\partial z} R_1 - P_2 \frac{\partial R_1}{\partial z}$$
$$+ \frac{\partial P_1}{\partial z} R_2 + P_1 \frac{\partial R_2}{\partial z}$$

$$= \left(Q_2 \frac{\partial P_1}{\partial y} + R_2 \frac{\partial P_1}{\partial z}\right) - \left(Q_1 \frac{\partial P_2}{\partial y} + R_1 \frac{\partial P_2}{\partial z}\right)$$
$$+ \left(\frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right) P_1 - \left(\frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) P_2$$

$$= \left(P_2 \frac{\partial P_1}{\partial x} + Q_2 \frac{\partial P_1}{\partial y} + R_2 \frac{\partial P_1}{\partial z}\right) - \left(P_1 \frac{\partial P_2}{\partial x} + Q_1 \frac{\partial P_2}{\partial y} + R_1 \frac{\partial P_2}{\partial z}\right)$$
$$+ \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right) P_1 - \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) P_2$$

Note that i-th components of

① $(F_2 \cdot \nabla) F_1 = \left(P_2 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y} + R_2 \frac{\partial}{\partial z}\right)(P_1)$

$$= P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}$$

② $(F_1\cdot\nabla)F_2 = P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}$

③ $(\nabla\cdot F_2)F_1 = \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)P_1$

④ $(\nabla\cdot F_1)F_2 = \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)P_2$.

So we get the result.

b) i-th component of $\nabla(F_1\cdot F_2)$

$$= \frac{\partial}{\partial x}(P_1P_2 + Q_1Q_2 + R_1R_2)$$

$$= P_1\frac{\partial P_2}{\partial x} + P_2\frac{\partial P_1}{\partial x} + Q_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial Q_1}{\partial x} + R_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial R_1}{\partial x}.$$

Note that the i-th components of

① $(F_1\cdot\nabla)F_2 = P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}$

② $(F_2\cdot\nabla)F_1 = P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}$

③ $F_1\times(\nabla\times F_2) = Q_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right) - R_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right)$

④ $F_2\times(\nabla\times F_1) = Q_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right) - R_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right)$

So we get the result.

28) Divergence thm implies

$$\iint_S \nabla f\cdot n\,dS = \iiint_D \nabla\cdot\nabla f\,dV = \iiint_D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)dV$$

$$= \iiint_D \left(\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2} + \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \right) dV$$

$$= \iiint_D \frac{1}{x^2+y^2+z^2} dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \frac{\rho^2 \sin\phi}{\rho^2} d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} a \sin\phi\, d\phi\, d\theta = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} a \sin\phi\, d\phi = \frac{\pi a}{2}$$ //

30) By ex.29, we have

$$\iint_S f \nabla g \cdot n\, dS = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV \quad — ①$$

$$\iint_S g \nabla f \cdot n\, dS = \iiint_D (g \nabla^2 f + \nabla g \cdot \nabla f)\, dV \quad — ②$$

① − ②, we get

$$\iint_S (f \nabla g - g \cdot \nabla f) \cdot n\, dS = \iiint_D (f \nabla^2 g - g \nabla^2 f)\, dV$$

since $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$.

Ex(6.3: 22) Let $F = \nabla f$.

Then
$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2+z^2} & — ① \\ \dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2+z^2} & — ② \\ \dfrac{\partial f}{\partial z} = \dfrac{2z}{x^2+y^2+z^2} & — ③ \end{cases}$$

From ①, $f = \ln(x^2+y^2+z^2) + g(y,z)$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2} + g_y(y,z)$$

(2) $\Rightarrow g_y(y,z) = 0, \quad g(y,z) = h(z)$

$$\Rightarrow f = \ln(x^2+y^2+z^2) + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2} + h'(z)$$

(3) $\Rightarrow h'(z) = 0, \quad h(z) = C.$

Hence $f(x,y,z) = \ln(x^2+y^2+z^2)$ is a potential function.

$$\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} F \cdot d\gamma = \ln(2^2+2^2+2^2) - \ln(1^2+1^2+1^2) = \ln 4 \,//$$

2b) $\begin{cases} \dfrac{\partial R}{\partial y} = \dfrac{-yz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial P}{\partial z} = \dfrac{-xz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \dfrac{\partial R}{\partial x} \\ \dfrac{\partial Q}{\partial x} = \dfrac{-xy}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \dfrac{\partial P}{\partial y} \end{cases}$

$\Rightarrow F$ is path independent.

3a) let $F = \nabla f$.

Then $\begin{cases} \dfrac{\partial f}{\partial x} = e^{yz} & \text{— (1)} \\ \dfrac{\partial f}{\partial y} = xze^{yz} + z\cos y & \text{— (2)} \\ \dfrac{\partial f}{\partial z} = xye^{yz} + \sin y & \text{— (3)} \end{cases}$

From ①, $f = xe^{yz} + g(y,z)$

$$\frac{\partial f}{\partial y} = xze^{yz} + g_y(y,z)$$

② $\Rightarrow g_y(y,z) = z\cos y \Rightarrow g(y,z) = z\sin y + h(z)$.

$$\Rightarrow f = xe^{yz} + z\sin y + h(z)$$

$$\frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z)$$

③ $\Rightarrow h'(z) = 0$, $h(z) = C$.

$\therefore f(x,y,z) = xe^{yz} + z\sin y$ is a potential function

As F is a gradient field, it's path independent.

So for a), b), c),

Work done $= f(1,\frac{\pi}{2},0) - f(1,0,1) = 0$

34) $g(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} F\,d\gamma = \int_{(0,0,0)}^{(x,y,z)} \nabla f\,d\gamma$

$\Rightarrow g(x,y,z) = f(x,y,z) - f(0,0,0)$.

$\Rightarrow \nabla g = \nabla f$.

38) a) Let $F = \nabla f$.

Then
$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{-GmMx}{(x^2+y^2+z^2)^{\frac{3}{2}}} & \text{— ①} \\ \dfrac{\partial f}{\partial y} = \dfrac{-GmMy}{(x^2+y^2+z^2)^{\frac{3}{2}}} & \text{— ②} \\ \dfrac{\partial f}{\partial z} = \dfrac{-GmMz}{(x^2+y^2+z^2)^{\frac{3}{2}}} & \text{— ③} \end{cases}$$

From ①, $f = \dfrac{GmM}{(x^2+y^2+z^2)^{\frac{1}{2}}} + g(y,z)$

$$\frac{\partial f}{\partial y} = \frac{-GmMy}{(x^2+y^2+z^2)^{\frac{3}{2}}} + g_y(y,z)$$

$$② \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$$

$$f = \frac{GmM}{(x^2+y^2+z^2)^{\frac{1}{2}}} + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{-GmMz}{(x^2+y^2+z^2)^{\frac{3}{2}}} + h'(z)$$

$$③ \Rightarrow h'(z) = 0, \quad h(z) = C.$$

So $f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ is a potential function.

b) Work done

$$= \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} F\, d\vec{r} = f(x_2,y_2,z_2) - f(x_1,y_1,z_1) = GmM\left(\frac{1}{S_2} - \frac{1}{S_1}\right),$$

where $S_i = (x_i^2 + y_i^2 + z_i^2)^{\frac{1}{2}}$ is the distance from (x_i, y_i, z_i) to origin, $i = 1, 2$.

Practice Problem:

Ex 16.8: 17) a) Let $G = (P, Q, R)$.

Then $\nabla \times G = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$

$\nabla \cdot (\nabla \times G) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0.$

if G is C^2.

b) By div. thm,

$$\iint_S (\nabla \times G) \cdot n\, dS = \iiint_D \nabla \cdot (\nabla \times G) \cdot n\, dV = 0.$$

18) a) Let $F_1 = (P_1, Q_1, R_1)$, $F_2 = (P_2, Q_2, R_2)$

$aF_1 + bF_2 = (aP_1 + bP_2,\ aQ_1 + bQ_2,\ aR_1 + bR_2).$

$$\nabla\cdot(aF_1+bF_2)=a\frac{\partial P_1}{\partial x}+b\frac{\partial P_2}{\partial x}+a\frac{\partial Q_1}{\partial y}+b\frac{\partial Q_2}{\partial y}+a\frac{\partial R_1}{\partial z}+b\frac{\partial R_2}{\partial z}$$

$$=a\left(\frac{\partial P_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)+b\left(\frac{\partial P_2}{\partial x}+\frac{\partial Q_2}{\partial y}+\frac{\partial R_2}{\partial z}\right)$$

$$=a\nabla\cdot F_1+b\nabla\cdot F_2$$

b) As determinant is linear,

$$\nabla\times(aF_1+bF_2)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ aP_1+bP_2 & aQ_1+bQ_2 & aR_1+bR_2 \end{vmatrix}$$

$$=a\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_1 & Q_1 & R_1 \end{vmatrix}+b\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_2 & Q_2 & R_2 \end{vmatrix}$$

$$=a\nabla\times F_1+b\nabla\times F_2$$

c) $F_1\times F_2=(Q_1R_2-Q_2R_1,-(P_1R_2-P_2R_1),P_1Q_2-P_2Q_1)$

$$\Rightarrow \nabla\cdot F_1\times F_2=\frac{\partial}{\partial x}(Q_1R_2-Q_2R_1)+\frac{\partial}{\partial y}(P_1R_2-P_2R_1)+\frac{\partial}{\partial z}(P_1Q_2-P_2Q_1)$$

$$=\left(Q_1\frac{\partial R_2}{\partial x}+R_2\frac{\partial Q_1}{\partial x}-Q_2\frac{\partial R_1}{\partial x}-R_1\frac{\partial Q_2}{\partial x}\right)$$

$$-\left(P_1\frac{\partial R_2}{\partial y}+R_2\frac{\partial P_1}{\partial y}-P_2\frac{\partial R_1}{\partial y}-R_1\frac{\partial P_2}{\partial y}\right)$$

$$+\left(P_1\frac{\partial Q_2}{\partial z}+Q_2\frac{\partial P_1}{\partial z}-P_2\frac{\partial Q_1}{\partial z}-Q_1\frac{\partial P_2}{\partial z}\right)$$

$$= P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)$$

$$+ P_1\left(\frac{\partial Q_2}{\partial z} - \frac{\partial R_2}{\partial y}\right) + Q_1\left(\frac{\partial R_2}{\partial x} - \frac{\partial P_2}{\partial z}\right) + R_1\left(\frac{\partial P_2}{\partial y} - \frac{\partial Q_2}{\partial x}\right)$$

$$= F_2 \cdot \nabla \times F_1 - F_1 \cdot \nabla \times F_2$$

19) a) $\nabla \cdot gF = \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gQ) + \frac{\partial}{\partial z}(gR)$

$$= \left(g\frac{\partial P}{\partial x} + P\frac{\partial g}{\partial x}\right) + \left(g\frac{\partial Q}{\partial y} + Q\frac{\partial g}{\partial y}\right)$$

$$+ \left(g\frac{\partial R}{\partial z} + R\frac{\partial g}{\partial z}\right)$$

$$= g\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} + R\frac{\partial g}{\partial z}\right)$$

$$= g\nabla \cdot F + \nabla g \cdot F$$

b) $\nabla \times (gF) = \left(\frac{\partial}{\partial y}(gR) - \frac{\partial}{\partial z}(gQ), \frac{\partial}{\partial z}(gP) - \frac{\partial}{\partial x}(gR), \frac{\partial}{\partial x}(gQ) - \frac{\partial}{\partial y}(gP)\right)$

$$= \begin{pmatrix} R\frac{\partial g}{\partial y} + g\frac{\partial R}{\partial y} - Q\frac{\partial g}{\partial z} - g\frac{\partial Q}{\partial z}, \\ P\frac{\partial g}{\partial z} + g\frac{\partial P}{\partial z} - R\frac{\partial g}{\partial x} - g\frac{\partial R}{\partial x}, \\ Q\frac{\partial g}{\partial x} + g\frac{\partial Q}{\partial x} - P\frac{\partial g}{\partial y} - g\frac{\partial P}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} g\frac{\partial R}{\partial y} - g\frac{\partial Q}{\partial z} \\ g\frac{\partial P}{\partial z} - g\frac{\partial R}{\partial x} \\ g\frac{\partial Q}{\partial x} - g\frac{\partial P}{\partial y} \end{pmatrix} + \begin{pmatrix} R\frac{\partial g}{\partial y} - Q\frac{\partial g}{\partial z} \\ P\frac{\partial g}{\partial z} - R\frac{\partial g}{\partial x} \\ Q\frac{\partial g}{\partial x} - P\frac{\partial g}{\partial y} \end{pmatrix}$$

$= g\nabla\times F + \nabla g\times F.$

25) See tutorial notes 7.

27) a) By div. thm,

$$\iint_S \nabla f\cdot n\,dS = \iiint_D \nabla^2 f\,dV = 0$$

b) By div. thm,

$$\iint_S f\nabla f\cdot n\,dS = \iiint_D \nabla\cdot(f\nabla f)\,dV.$$

By ex. 19 a), $\nabla\cdot(f\nabla f) = f\nabla^2 f + \nabla f\cdot\nabla f = |\nabla f|^2$

Hence $\iint_S (f\nabla f\cdot n)\,dS = \iiint_D |\nabla f|^2\,dV.$

29) By div. thm,

$$\iint_S f\nabla g\cdot n\,dS = \iiint_D \nabla\cdot(f\nabla g)\,dV$$

By ex. 19 a), $\nabla\cdot(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g.$

Hence $\iint_S (f\nabla g\cdot n)\,dS = \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV.$

31) a) $\iiint_D p(t,x,y,z)\,dV$ represents the mass of the fluid at any time t.

The equation says that the rate of change of mass is equal to the flux across the surface S of D.

If $\vec{v}$ is in the same direction as $\vec{n}$, then the fluid flow out and the mass decreases.

Similarly, if ∇B in opposite direction as $\vec{n}$, then the fluid flow in and the mass increases.

b) $\iiint_D \frac{\partial \rho}{\partial t} dV = \frac{d}{dt} \iiint_D \rho \, dV = -\iint_S \rho v \cdot n \, dS = -\iiint_D \nabla \cdot \rho v \, dV$

$$\Rightarrow \iiint_D \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v \right) dV = 0 \text{ for any } D.$$

Claim: If $\iiint_D f \, dV = 0$ for any D, then $f \equiv 0$.

Pf: Suppose not.
WLOG, assume $f(x_0, y_0, z_0) > 0$ for some $(x_0, y_0, z_0) \in \mathbb{R}^3$.
By continuity, $f(x, y, z) > 0$ for a sufficiently small ball B centred at (x_0, y_0, z_0).

$$\Rightarrow \iiint_B f \, dV > 0, \text{ contradiction } \boxed{}$$

By claim, we have $\nabla \cdot \rho v + \frac{\partial \rho}{\partial t} = 0$.

32) a) ∇T points in the direction of max. increase of temperature.
If the solid is heating up at a pt. P, then ∇T at the point surrounding P will point to P.
As heat flows from high temperature to low temperature, $-\nabla T$ points in the direction of heat flow.

b) By ex 31 b),

$$\frac{\partial}{\partial t}(c \rho T) = -\nabla \cdot \rho v = \nabla (k \nabla T) = k \nabla^2 T$$

$$\Rightarrow \frac{\partial}{\partial t} T = \frac{k}{c\rho} \nabla^2 T = K \nabla^2 T.$$